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EXISTENCE AND STABILTY OF MULTIDIMENSIONAL TRAVELLING WAVES IN THE MONOSTABLE CASE

BY

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ABSTRACT

The paper is devoted to the existence and stability of travelling waves described by monotone parabolic systems in the monostable case. The first initial-boundary value problem is considered. Existence of waves is proved for all values of the velocity greater than or equal to the minimal velocity. A minimax representation for the minimal velocity is obtained. Stability of monotone waves in a weighted norm is proved.

1. Introduction

We consider the parabolic system of equations

$$\frac{\partial u}{\partial t} = a(x')\Delta u + \sum_{j=1}^{m} b_j(x')\frac{\partial u}{\partial x_j} + F(u, x'), \tag{1.1}$$

in a cylinder $\Omega = \Omega' \times R$. Here $u(x) = (u_1(x), ..., u_n(x))$, $x = (x_1, ..., x_m)$, x_1 is the variable along the axis of the cylinder, $x' = (x_2, ..., x_m)$ is the variable in the

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section Ω' of the cylinder, the domain Ω' is bounded and has a boundary of the class $C^{2+\delta}$ with some positive δ, $a(x')$, $b_j(x')$ are smooth diagonal matrices,

$$a_i(x') \geq a_0 > 0, \quad x' \in \Omega', \quad i = 1, ..., n, \quad j = 1, ..., m,$$

where a_i are diagonal elements of the matrix a, $F = (F_1, ..., F_n)$ is a smooth vector-valued function. On the boundary $\partial\Omega$ of the cylinder we consider the boundary conditions

$$u(x) = \phi(x'), \quad x \in \partial\Omega, \tag{1.2}$$

where $\phi(x') \in C^{2+\delta}(\bar{\Omega})$ depends only on the variable in the section of the cylinder.

The travelling wave solution of the problem (1.1), (1.2) is a solution of the form

$$u(x,t) = w(x_1 - ct, x_2, ..., x_m),$$

where c is an unknown constant, the wave velocity. This function is a solution of the problem

$$a(x')\Delta w + c\frac{\partial w}{\partial x_1} + \sum_{j=1}^{m} b_j(x')\frac{\partial w}{\partial x_j} + F(w, x') = 0, \tag{1.3}$$
$$w(x) = \phi(x'), \quad x \in \partial\Omega.$$

We assume that for $n > 1$ the nonlinearity F satisfies the following condition:

$$\frac{\partial F_i}{\partial u_j} \geq 0, \quad i \neq j, \quad i, j = 1, \ldots, n.$$

This condition means that we can use comparison theorems for the systems under consideration. For $n = 1$ there are no additional conditions and the comparison theorems are also applicable. The systems of this type arise in numerous applications (see [15]).

We look for the travelling waves having limits at infinity:

$$\lim_{x_1 \to \pm\infty} w(x) = w_\pm(x'), \tag{1.4}$$

where the functions $w_\pm$ are solutions of the problem in the section of the cylinder:

$$a(x')\Delta' w_\pm + \sum_{j=2}^{m} b_i(x')\frac{\partial w_\pm}{\partial x_j} + F(w_\pm, x') = 0, \tag{1.5}$$

$$w_\pm(x') = \phi(x'), \quad x' \in \partial\Omega', \tag{1.6}$$

where Δ' is the Laplace operator with respect to the variables in the section of the cylinder.

We recall the classification of the problems according to stability of solutions w_+ and w_-. We consider the eigenvalue problem for the corresponding linearized equation

$$L^{\pm}u \equiv a(x')\Delta' u + \sum_{j=2}^{m} b_j(x')\frac{\partial u}{\partial x_j} + F'(w_{\pm}(x'), x')u = \lambda u \tag{1.7}$$

$$u = 0, \quad x' \in \partial\Omega'. \tag{1.8}$$

If all eigenvalues of both operators L^+ and L^- are in the left half-plane, then it is the so-called bistable case. If for one of them there are eigenvalues in the right half-plane and for another one all eigenvalues have negative real parts, it is the monostable case. Finally, in the unstable case both operators have eigenvalues in the right half-plane. As is well known (see [15]) properties of travelling waves are different in these three cases.

In this work we study the monostable case. We suppose that all eigenvalues of the operator L^- have negative real parts, and there are eigenvalues of the operator L^+ with positive real parts. For the one-dimensional scalar equation ($m = n = 1$) this problem was first studied in [7] and then followed by a large number of other works (see [15] for the references). For the multidimensional scalar equation the problem was considered in [2]–[4], [6], [14] and for one-dimensional systems of equations in [15], [17]. In this work we consider multidimensional systems of equations and develop the methods and results suggested in our previous works for one-dimensional systems.

Proof of the existence of travelling waves is based on the comparison theorems which are valid for the systems under consideration. We first prove existence of solutions of some auxiliary problems in half-cylinders and then, passing to the limt, we obtain existence of solutions in the whole cylinder.

Stability of travellig waves with respect to small perturbations is based on the analysis of the spectrum of the linearized operator which can have independent interest. Consider the linear operator

$$Lu = a(x)\Delta u + \sum_{j=1}^{m} b_j(x)\frac{\partial u}{\partial x_j} + c(x)u$$

assuming that the matrices a, b_j, c are sufficiently smooth, a and b_j are diagonal, c has nonnegative off-diagonal elements and is functionally irreducible. We consider this operator on functions $u \in C^{2+\delta}(\bar{\Omega})$ satisfying the boundary condition $u = 0$

on $\partial\Omega$. Denote σ_e the supremum of the real parts of the points of the essential spectrum of this operator, and σ_d of its discrete spectrum. In [18] we showed that if $\sigma_d > \sigma_e$, then the principal eigenvalue of the operator is real, simple, and the corresponding eigenfunction is positive. Since the derivative of the monotone in x_1 wave is an eigenfunction corresponding to zero eigenvalue, then it allows us to conclude that all other eigenvalues are in the left half-plane and to prove the stability of the monotone wave.

However, we can make this conclusion only in the case if the essential spectrum lies in the left half-plane, i.e. $\sigma_e < \sigma_d$. This inequality holds in the bistable case, but not in the monostable case considered in this work. In the monostable case $\sigma_e > \sigma_d$, and to move the essential spectrum to the left half-plane we introduce a weighted norm. If the derivative of the wave belongs to this weighted space, we can still use the result above. So the question is what happens if it does not belong to this space. In other words, we have a positive solution v of the equation $Lu = 0$ which should be understood as a local equality because the function v may not belong to $C^{2+\delta}(\bar{\Omega})$ and may even be unbounded. We show that in this case all eigenvalues of the operator lie in the left half-plane. It is a generalization of the results obtained in our previous work [17] for one-dimensional systems.

The contents of the paper are as follows. In Section 2 we prove existence of waves for all values of velocities greater than or equal to the minimal velocity c_0. We obtain a miminax representation for the minimal velocity and show that there are no waves with velocity less than c_0. In Section 3 we prove stability of waves in a weighted norm.

2. Existence of waves

2.1 MAIN THEOREM. We make the following assumptions on the solutions w_+ and w_- of the problem (1.5), (1.6).

Assumption 1: The functions $w_\pm$ belong to $C^{(2+\delta)}(\bar{\Omega}')$ and the following inequality

$$w_+(x') < w_-(x'), \quad x' \in \Omega' \tag{2.1}$$

holds.

Assumption 2: There are no other solutions of the problem (1.5), (1.6) satisfying the inequality

$$w_+(x') \le w \le w_-(x'), \quad x' \in \Omega'. \tag{2.2}$$

Assumption 3: There exists a sequence of functions $\{v_n(x')\}$ uniformly bounded in $C^{(2+\delta)}(\bar{\Omega}')$ satisfying the inequality (2.2) and the following conditions:

$$v_n(x') \to w_+(x') \quad \text{as } n \to \infty \tag{2.3}$$

uniformly in x',

$$a(x')\Delta' v_n + \sum_{j=2}^{m} b_j(x')\frac{\partial v_n}{\partial x_j} + F(v_n, x') \geq 0, \tag{2.4}$$

$$v_n(x') = \phi(x'), \quad x' \in \partial\Omega'. \tag{2.5}$$

The last assumption can be formulated in terms of eigenvalues of the linearized problem. We discuss it below.

We prove first the theorem on wave existence under Assumptions 1–3. We will show below that Assumption 2 is not always necessary and can be replaced by a weaker assumption.

THEOREM 2.1: *Let Assumptions 1–3 be satisfied. Then there exists a constant c_0 such that for every $c \geq c_0$ there exists a monotone in x_1 solution of the problem* (1.3), (1.4). *The constant c_0 is given by the minimax representation*

$$c_0 = \inf_{\rho \in K} \sup_{x \in \Omega,\ i=1,\dots,n} B_i, \tag{2.6}$$

where

$$B_i = \Big(a_i(x')\Delta\rho_i + \sum_{j=1}^{m} b_{ij}(x')\frac{\partial \rho_i}{\partial x_j} + F_i(\rho, x') \Big) \Big/ \Big(-\frac{\partial \rho_i}{\partial x_1} \Big), \tag{2.7}$$

b_{ij} are diagonal elements of the matrix b_j, K is a class of sufficiently smooth vector-valued functions $\rho(x)$ such that

$$\lim_{x_1 \to \pm\infty} \rho(x) = w_\pm(x'), \quad \rho(x) = \phi(x') \quad \text{for } x \in \partial\Omega, \tag{2.8}$$

the derivative $\partial\rho/\partial x_1$ is negative and the normal derivative $\partial(\rho - w_+(x))/\partial\nu$, $x \in \partial\Omega$ in the direction of the outer normal is also negative.

For $c < c_0$ such solutions do not exist.

Remark: We show below that there exists a function $\rho(x)$ such that B_i is bounded and that the minimal value of the velocity given by (2.6) is finite.

Proof: We take $c > c_0$. Then there exists a function ρ for which $\sup_{x,i} B_i < c$. It means that

$$a(x')\Delta\rho + c\frac{\partial \rho}{\partial x_1} + \sum_{j=1}^{m} b_j(x')\frac{\partial \rho}{\partial x_j} + F(\rho, x') < 0 \tag{2.9}$$

for all $x \in \Omega$. For any given N we can choose a function $v_k(x')$ which satisfies Assumption 3 and such that

$$w_+(x') \le v_k(x') \le \rho(x)\,|_{x_1=N}\,. \tag{2.10}$$

Indeed, since $\rho \to w_+$ as $x_1 \to \infty$ and the derivative $\partial\rho/\partial x_1$ is negative, then for all internal points x' we have the inequality

$$w_+(x') < \rho(x)\,|_{x_1=N}\,. \tag{2.11}$$

On the other hand, $v_k \to w_+$ as $k \to \infty$. Then for k sufficiently large the inequality (2.10) holds in any closed domain $\Omega_0' \subset \Omega'$.

From the convergence (2.3) and the uniform boundedness of the second derivatives of the functions v_k there follows the convergence of the first derivatives:

$$\frac{\partial v_k}{\partial x_i} - \frac{\partial w_+}{\partial x_i} \to 0, \quad i = 1, \ldots, m \tag{2.12}$$

uniformly in x'. Since the derivative $\partial(\rho - w_+(x'))/\partial\nu$, $x_1 = N$ is negative, we obtain the inequality (2.10) for all $x' \in \Omega'$ and sufficiently large k.

We introduce a domain $\Omega_N \subset \Omega$ located in the half-space $\{x_1 \le N\}$ and such that $\Omega_N = \Omega$ for $x_1 \le N - 1$, and the boundary $\partial\Omega_N$ belongs to $C^{(2+\delta)}$. We consider now the initial-boundary value problem for the equation

$$\frac{\partial u}{\partial t} = \Delta u + a^{-1}(x')\left(c\frac{\partial u}{\partial x_1} + \sum_{j=1}^{m} b_j(x')\frac{\partial u}{\partial x_j} + F(u,x') - f(x')e^{-t}\right) \tag{2.13}$$

in the domain Ω_N with the initial condition

$$u(x,0) = v_k(x'), \tag{2.14}$$

and the boundary conditions

$$u(x,t)\,|_{\partial\Omega_N} = v_k(x'). \tag{2.15}$$

Here

$$f(x') = a(x')\Delta' v_k + \sum_{j=2}^{m} b_j(x')\frac{\partial v_k}{\partial x_j} + F(v_k, x').$$

To show existence of the classical solution of this problem, we construct a sequence of bounded domains Ω_N^i, $i = 1, 2, \ldots$ with sufficiently smooth boundary and such that

$$\Omega_N^i \subset \Omega_N^{i+1}, \quad \bigcup_{i=1}^{\infty} \Omega_N^i = \Omega_N$$

and consider equation (2.13) with initial condition (2.14) in the domain Ω_N^i with the boundary condition

$$u(x,t)\mid_{\partial\Omega_N^i}=v_k(x^{'}). \tag{2.16}$$

We note that the compatibility condition for the problem (2.13), (2.14), (2.16) is satisfied. So the classical solution exists (see [8]).

We show that the solution of the problem (2.13), (2.14), (2.16) satisfies the inequality

$$u(x,t)\le\rho(x).$$

Indeed, the function

$$z(x,t)=u(x,t)-\rho(x)$$

satisfies the equation

$$\frac{\partial z}{\partial t}=\Delta z+a^{-1}(x^{'})\Big(c\frac{\partial z}{\partial x_1}+\sum_{j=1}^{m}b_j(x^{'})\frac{\partial z}{\partial x_j}+F(u,x^{'})-F(\rho,x^{'})-f(x^{'})e^{-t}-g\Big)$$

where

$$g(x)=-\Big(a(x^{'})\Delta\rho+c\frac{\partial\rho}{\partial x_1}+\sum_{j=1}^{m}b_j(x^{'})\frac{\partial\rho}{\partial x_j}+F(\rho,x^{'})\Big)>0.$$

Since $z(x,0)\le 0$, $z(x,t)\mid_{\partial\Omega_N^i}\le 0$, $f(x^{'})$ and $g(x)$ are nonnegative, then $z(x,t)\le 0$ for all $t\ge 0$ and $x\in\Omega_N^i$.

We note that the solution of the problem (2.13), (2.14), (2.16) increases in time. Indeed, the function $v=\partial u/\partial t$ satisfies the problem

$$\frac{\partial v}{\partial t}=\Delta v+a^{-1}\left(c\frac{\partial v}{\partial x_1}+\sum_{k=1}^{n}b_k\frac{\partial v}{\partial x_k}+F^{'}(u,x^{'})v+fe^{-t}\right),$$
$$v\mid_{t=0}=0,\quad v\mid_{\partial\Omega_N^i}=0.$$

Then $v(x,t)\ge 0$ and $u(x,t)\ge v_k(x^{'})$.

So the solution $u_N^i(x,t)$ of the problem (2.13), (2.14), (2.16) is bounded independently of i in C and, consequently, in the $C^{(2+\delta)}$-norm. Choosing a converging subsequence, we obtain a classical solution $u_N(x,t)$ of the problem (2.13)–(2.15). As before we show that

$$w_+(x^{'})\le v_k(x^{'})\le u_N(x,t)\le\rho(x)$$

and $u_N(x,t)$ is increasing in t. Passing to the limit as $t \to \infty$, we obtain a solution $w_N(x)$ of the equation (1.3) in the domain Ω_N with the boundary condition

$$w_N \mid_{\partial\Omega \cap \partial\Omega_N} = \phi(x').$$

It is bounded in the $C^{(2+\delta)}$-norm independently of N. Moreover

$$w_+(x') \leq w_N(x) \leq \rho(x).$$

We note also that $\partial w_N/\partial x_1 \leq 0$. Indeed, the function $z = \partial u_N/\partial x_1$ satisfies the problem

$$\frac{\partial z}{\partial t} = \Delta z + a^{-1}\left(c\frac{\partial z}{\partial x_1} + \sum_{k=1}^{n} b_k \frac{\partial z}{\partial x_k} + F'(u,x')z \right),$$
$$z|_{\partial\Omega} = 0, \quad z|_{t=0} = 0,$$

and $z|_{\partial\Omega_N} \leq 0$ since $u_N(x,t) \geq v(x')$. Then $z \leq 0$.

Since the function $w_N(x)$ is monotone in x_1, then there exists the limit

$$w_N^-(x') = \lim_{x_1 \to -\infty} w_N(x).$$

It satisfies the problem (1.5), (1.6) and the inequality

$$v_k(x') \leq w_N^-(x') \leq w_-(x').$$

Then $w_N^- = w_-$.

Thus we have constructed the sequence of functions $\{w_N\}$, $N = 1, 2, \ldots$ defined for $x_1 \leq N$. We show now how to construct the solution in the whole cylinder Ω.

We choose an arbitrary interior point in Ω. Without loss of generality we can assume that it is $x = 0$. Denote $x_N = (x_{1N}, 0^{n-1})$, where 0^{n-1} is a $(n-1)$-vector with zero elements, x_{1N} is a solution of the equation

$$w_{N1}(x_1, 0^{n-1}) = \rho_1(0, 0^{n-1}).$$

Here the index 1 shows the number of the component of the vector-valued function. Existence and uniqueness of the solution follow from the monotonicity of the function w_N in x_1. Denote further $\tilde{w_N}(x) = w_N(x + x_N)$ so that $\tilde{w_N}_1(0) = \rho_1(0)$.

Since $w_{N1}(0) \leq \rho_1(0)$, then $x_N \leq 0$ and the functions $\tilde{w_N}$ are defined at least for $x_1 \leq N$. Since the derivatives of these functions are uniformly bounded, then for any given M from the sequence $\{\tilde{w_N}\}$ we can choose a subsequence converging to some limiting function uniformly on the set $\Omega \cap \{[-M, M]\}$. If we consider now a sequence of M going to infinity and choosing each time a converging subsequence, we obtain the limiting function $w(x)$ defined in the whole Ω. It is a solution of equation (1.3) as a limit of solutions of this equation, and it satisfies the boundary conditions (1.2). The function $w(x)$ has a nonnegative derivative with respect to x_1. Hence there exist the limits of $w(x)$ as $x_1 \to \pm\infty$. We have

$$w_{+1}(0) < w_{N1}(0) = \rho_1(0) < w_{-1}(0).$$

Since the limits $\lim_{x_1\to\pm\infty} w(x)$ satisfy the problem (1.5), (1.6), then we necessarily obtain (1.4).

We have already mentioned that the derivative $\partial w/\partial x_1$ is not positive. We show now that it is strictly negative assuming that the matrix $\partial F_i/\partial w_j$ is functionally irreducible. Indeed, the function $y = \partial w/\partial x_1$ is a solution of the problem

$$a(x')\Delta y + c\frac{\partial y_i}{\partial x_1} + \sum_{j=1}^{m} b_{ij}(x')\frac{\partial y_i}{\partial x_j} + \sum_{j=1}^{n} \frac{\partial F_i(w(x)))}{\partial w_j} y_i = 0,$$

$$y|_{\partial\Omega} = 0,$$

and $y \geq 0$. By the positivity theorems $y_k > 0$ in Ω or $y_k \equiv 0$. The latter cannot take place because $w(x)$ has different values at $x_1 = \pm\infty$.

Thus the theorem is proved for $c > c_0$.

To show existence of a travelling wave for $c = c_0$ we consider a sequence $\{c_n\}$, $c_n > c_0$, converging to c_0. From the corresponding sequence of waves $\{w^{(n)}\}$ with the velocities c_n we can choose a subsequence converging to some limiting function w_0. It satisfies equation (1.3) with $c = c_0$ and the boundary conditions.

Consider finally the case $c < c_0$. Suppose that there is a solution $w(x)$ of the problem (1.3), (1.4) monotone in x_1. By virtue of the maximum principle the normal derivative $\partial(w - w_+)/\partial n$ on the boundary of the cylinder is negative, and $w \in K$. Then

$$c_0 > \inf_{\rho\in K} \sup_{x,i} B_i.$$

This contradiction proves the theorem. ∎

We note that Assumption 3 means that the principal eigenvalue of the operator L^+ given by (1.7), (1.8) is nonnegative. If we suppose that it is positive, then we can put

$$v_n(x') = w_+(x') + \tau_n u_0(x'),$$

where $\tau_n > 0$ and $\tau_n \to 0$ as $n \to \infty$; u_0 is an eigenfunction corresponding to the principal eigenvalue. Since $u_0(x') > 0$, $x' \in \Omega'$, then the functions v_n defined in this way satisfy Assumption 3.

2.2 EXAMPLE OF A TEST FUNCTION. We present here an example of a test function for which the expressions B_i given by (2.7) are bounded. It will show that the minimal velocity is bounded and the waves exist.

We put

$$\rho(x) = w_+(x') + \psi(x_1)(w_-(x') - w_+(x')),$$

where $\psi'(x_1) < 0$, $\psi(-\infty) = 1$, $\psi(\infty) = 0$. Then

$$-B_i = a_1(x')\frac{\psi''}{\psi'} + b_{1i}(x')$$

$$+\frac{\psi[F_i(w_+,x') - F_i(w_-,x')] + F_i(w_+ + \psi(w_- - w_+),x') - F_i(w_+,x')}{\psi'(w_- - w_+)_i}. \tag{2.17}$$

We can choose the function $\psi(x)$ such that ψ''/ψ' is bounded for all x, ψ/ψ' is bounded at $+\infty$ and $(1-\psi)/\psi'$ at $-\infty$. This means that it approaches exponentially its limits at infinity. We show that the third summand in the right hand side of (2.17) remains bounded.

We note first that the expression $(w_- - w_+)_i/(w_- - w_+)_j$ is bounded for $x' \in \bar{\Omega}'$. Indeed, in the interior points of Ω' it follows from the inequality $w_- > w_+$. On the boundary

$$\frac{\partial w_-}{\partial \nu} > \frac{\partial w_+}{\partial \nu}.$$

As $x' \to \partial\Omega$, the numerator of the last term in (2.17) has the form

$$\psi(B(x)(w_+ - w_-))_i + (\tilde{B}(x)\psi(w_- - w_+))_i,$$

where

$$B(x) = \int_0^1 F'(w_+ + t(w_- - w_+), x')dt, \quad \tilde{B}(x) = \int_0^1 F'(w_+ + t\psi(w_- - w_+), x')dt.$$

When $x_1 \to +\infty$, we represent it in the form

$$\psi[F_i(w_+,x') - F_i(w_-,x')] + \psi(\tilde{B}(x)(w_+ - w_-))_i.$$

Finally, as $x_1 \to -\infty$, we can consider it in the form

$$(1-\psi)[-F_i(w_+,x') + F_i(w_-,x')] + (1-\psi)(\hat{B}(x)(w_+ - w_-))_i,$$

where

$$\hat{B}(x) = \int_0^1 F'(w_+ + \psi(w_- - w_+) + t(1-\psi)(w_+ - w_-))dt.$$

In all three cases, divided by $\psi'(w_- - w_+)$, it remains bounded.

Thus we have shown that c_0 in the theorem is bounded from above, and the waves exist. Its boundedness from below follows from Theorem 2.2 (see also (2.19)).

2.3 SIGN OF VELOCITY. We show that if the principal eigenvalue of the operator L^+ is positive, then the wave velocity is positive.

Consider the operator

$$A(v) = a(x')\Delta' v + \sum_{k=2}^{m} b_k(x')\frac{\partial v}{\partial x_k} + F(v, x')$$

acting on the functions from $C^{(2+\delta)}(\bar{\Omega}')$ with the boundary condition

$$v(x') = \phi(x'), \quad x' \in \partial\Omega'.$$

Let

$$v(x') = w_+(x') + u(x'),$$

where

$$u(x') \geq 0, \quad x' \in \bar{\Omega}', \quad u(x') = 0, \quad x' \in \partial\Omega'.$$

Then

$$A(v) = L^+ u + B(x')u,$$

where

$$B(x') = \int_0^1 F_v'(w_+ + tu, x')dt - F_v'(w_+, x').$$

Denote λ_0 the principal eigenvalue of the operator L^+ and $u^*(x')$ the corresponding eigenfunction of the adjoint operator L^{+*}. Then

$$J \equiv \int_{\Omega'} (u^*, A(v))dx' = \lambda_0 \int_{\Omega'} (u^*, u)dx' + \int_{\Omega'} (u^*, B(x')u)dx'.$$

By assumption, λ_0 is positive. The eigenfunction u^* is positive in Ω' if $F'(w_+(x'), x')$ is functionally irreducible. If we assume that the derivative F_v' satisfies the Lipschitz condition, then for any function u positive in Ω' and sufficiently small in C-norm, $J > 0$.

Equation (1.3) can be represented in the form

$$a\frac{\partial^2 w}{\partial x_1^2} + (c + b_1)\frac{\partial w}{\partial x_1} + A(w) = 0.$$

Denote

$$z(x) = w(x) - w_+(x').$$

Then

$$a\frac{\partial^2 z}{\partial x_1^2} + (c+b_1)\frac{\partial z}{\partial x_1} + A(w) = 0.$$

Multiplying this equation by u^* and integrating over Ω', we obtain

$$\int_{\Omega'} \left(u^*, a\frac{\partial^2 z}{\partial x_1^2}\right) dx' + \int_{\Omega'} \left(u^*, (c+b_1)\frac{\partial z}{\partial x_1}\right) dx' + \int_{\Omega'} (u^*, A(w))dx' = 0.$$

Since $w(x) \to w_+(x')$ as $x_1 \to +\infty$ uniformly in x', then there exists x_1^0 such that the last integral in the left hand side of this equality is positive for all $x_1 \geq x_1^0$. Let $x_1^1 \geq x_1^0$. We integrate the last equality from x_1^0 to x_1^1:

$$\int_{\Omega'} \left(u^*, a\frac{\partial z}{\partial x_1}\right)\bigg|_{x_1^1} dx' - \int_{\Omega'} \left(u^*, a\frac{\partial z}{\partial x_1}\right)\bigg|_{x_1^0} dx' + \int_{\Omega'} (u^*, (c+b_1)z)|_{x_1^1} dx'$$

$$\text{(2.18)} \qquad - \int_{\Omega'} (u^*, (c+b_1)z)|_{x_1^0} dx' + \int_{x_1^0}^{x_1^1} \int_{\Omega'} (u^*, A(w))dx' = 0.$$

We note that the second derivative $\partial^2 z/\partial x_1^2$ is bounded. Since $z(x) \to 0$ as $x_1 \to +\infty$, then the first integral in the left hand side of (2.18) tends to zero, and the last integral increases and tends to a finite limit. Passing to the limit in (2.18) as $x_1 \to +\infty$, we obtain

$$\text{(2.19)} \qquad \int_{\Omega'} (u^*, (c+b_1)z)\Big|_{x_1^0} dx' = - \int_{\Omega'} \left(u^*, a\frac{\partial z}{\partial x_1}\right)\bigg|_{x_1^0} dx' + \int_{x_1^0}^{\infty} \int_{\Omega'} (u^*, A(w))dx'.$$

The right hand side of the last equality is positive. Then $c + b_1(x') \not\equiv 0$ and, if $c + b_1(x') \neq 0$, then $c + b_1(x') > 0$.

We have proved the following theorem.

THEOREM 2.2: *Suppose that the principal eigenvalue of the operator L^+ is positive. Then for any wave monotone in x_1, $c+b_1(x') \not\equiv 0$ and, if $c+b_1(x') \neq 0$ for all $x' \in \Omega'$, then $c + b_1(x') > 0$. If $b_1 \equiv 0$, then c is positive.*

COROLLARY 1: *Let $c_0(b_1)$ denote the value of the minimal velocity as a functional determined by the coefficient $b_1(x')$. If*

$$c_0(0) \geq \max_x b_1(x') - \min_x b_1(x'),$$

then

$$c_0(b_1) + b_1(x') > 0.$$

COROLLARY 2: *Suppose that $b_1 \equiv 0$ and the principal eigenvalues of both operators L^+ and L^- are positive. Then monotone waves do not exist.*

Proof: If a monotone wave existed, we could repeat the same construction for $x_1 \to -\infty$ and obtain $c < 0$.

Remarks: (1) The last assertion shows that monotone waves do not exist in the unstable case. The sign of the wave velocity is determined by local properties of the operators in small neighbourhoods of w_+ and w_-. So this results remains valid for wider classes of equations than we consider in this work (cf. [15], [17]).

(2) In the case $b_1 \equiv 0$, Theorem 2.1 remains valid if instead of Assumption 2 we require that there is a finite number of solutions of the problem (1.5), (1.6) satisfying (2.2) and, for all of them except w_-, the corresponding linearized operators have eigenvalues in the right half-plane.

3. Stability of waves

3.1 SPECTRUM AND STABILITY. We begin with some general results on stability of stationary solutions in Banach spaces. Their proofs are given in [15], [16]. In the next subsection we use them to prove stability of travelling waves.

We consider the equation

$$\frac{du}{dt} = Au + f(u) \tag{3.1}$$

in a Banach space E. Here $u(t) \in E$ for all $t \in [0, \infty)$, A is a linear and f a nonlinear operator, acting in E.

Let the stationary equation

$$Au + f(u) = 0 \tag{3.2}$$

have a family of solutions $u = \phi_\alpha \in E$. Here α is a real parameter, $\alpha \in (-\bar{\alpha}, \bar{\alpha})$. We present the results on the stability of the stationary solutions with respect to small perturbations in an arbitrary Banach space $H \subset E$.

Let the following conditions on ϕ_α, $f(u)$, and A be satisfied:

Assumption 1: (a) There exists the derivative ϕ'_α of the stationary solution with respect to α, $\alpha \in (-\bar{\alpha}, \bar{\alpha})$, taken in the norm of the space H, $\phi'_\alpha \in H$ (it is not assumed here that $\phi_\alpha \in H$).

(b) ϕ'_α satisfies the Lipchitz condition in $\alpha, \alpha \in (-\bar{\alpha}, \bar{\alpha})$ in the norm of the space H.

Assumption 2: (a) Nonlinear operator $f(u)$ is defined on the whole E, is bounded and has the first Gâteaux differential $f'(u)$ in an arbitrary direction v in the space E, $u, v, f'(u)v \in E$.

(b) The Gâteaux differential $f'(u)v$ is continuous in $u \in E$ for any fixed $v \in E$.

(c) The operator $f'(\phi_\alpha + v)$ is bounded in H for $v \in H$, $\alpha \in (-\bar{\alpha}, \bar{\alpha})$ and satisfies the Lipschitz condition in v for $\|v\| \leq 1$. Here $\|\cdot\|$ is the norm in H.

Assumption 3: (a) There is a restriction of the operator A, acting in H, which is a generator of an analytical semi-group.

(b) The spectrum $\sigma(L)$ of the operator $L = A + f'(\phi_0)$ is as follows: zero is a simple eigenvalue; all other points of the spectrum lie in a closed angle situated in the left half of the complex plane. This means that there are positive numbers a_1 and b_1 such that

$$\mathrm{Re}\,\lambda + a_1 \mid \mathrm{Im}\,\lambda \mid + b_1 \leq 0 \tag{3.3}$$

for $\lambda \in \sigma(L)$, $\lambda \neq 0$.

THEOREM 3.1: *Let Assumptions 1–3 be satisfied. Then there is a positive ϵ such that for any $\bar{u} \in E$ satisfying the condition $\mid \bar{u} - \phi_0 \mid \leq \epsilon$, the solution $u(t)$ of equation (3.1) with initial condition $u(0) = \bar{u}$ exists in the space E for any $t \in [0, \infty)$. This solution is unique, and for some $\alpha \in (-\bar{\alpha}, \bar{\alpha})$ the estimation*

$$\|u(t) - \phi_\alpha\| \leq M e^{-bt}$$

is valid. Here b and M do not depend on $\bar{u}$, α, and t, $b > 0$.

Remark: It is asssumed in the formulation of the Theorem that $\bar{u} - \phi_0 \in H$, but it is not assumed that $\bar{u}$, $\phi_0 \in H$. ∎

We need also the stability theorem for the case when the stationary equation (3.2) has an isolated solution ϕ_0. Let Assumption 2 with $\alpha = 0$ and Assumption 3(a) be satisfied, and (3.3) be satisfied for all $\lambda \in \sigma(L)$. Then the following well-known result occurs:

THEOREM 3.2: *There is a positive ϵ such that for any $\bar{u} \in E$, satisfying the condition $\|\bar{u} - \phi_0\| \leq \epsilon$, the solution $u(t)$ of equation (3.1) with initial condition $u(0) = \bar{u}$ exists in the space E for any $t \in [0, \infty)$. This solution is unique and the estimation*

$$\|u(t) - \phi(t)\| \leq M e^{-bt}$$

is valid. Here b is an arbitrary number less than b_1; M does not depend on $\bar{u}$ and t.

3.2 Local stability of waves. In Section 2 we have shown existence of travelling waves, i.e. stationary solutions of the equation

$$\frac{\partial u}{\partial t} = a(x')\Delta u + c\frac{\partial u}{\partial x_1} + \sum_{j=1}^{m} b_j(x')\frac{\partial u}{\partial x_j} + F(u,x'), \tag{3.4}$$

in the cylinder Ω with a bounded section Ω', with boundary condition

$$u(x) = \phi(x'), \quad x \in \partial\Omega. \tag{3.5}$$

We have proved also that these waves are monotone in x_1. We apply now the results of Section 3.1 to prove local stability of waves monotone in x_1.

We introduce spaces E and H. We put $E = C(\bar{\Omega})$. Let $L^2_\mu(\Omega)$ denote the weighted L^2 space with norm

$$\|u\|_{L^2_\mu} = \|\mu u\|_{L^2}.$$

$C_0(\bar{\Omega})$ is a space of continuous functions which satisfy the boundary condition $u = 0$, $x \in \partial\Omega$ and tend to 0 as $|x| \to \infty$ and $C_{0,\mu}(\bar{\Omega})$ is a weighted space, $u \in C_{0,\mu}(\bar{\Omega})$ if $\mu u \in C_0(\bar{\Omega})$. Consider the spaces

$$H^0 = L^2(\Omega) \cap C_0(\bar{\Omega}) \quad \text{and} \quad H = L^2_\mu(\Omega) \cap C_{0,\mu}(\bar{\Omega})$$

with norms

$$\|u\|_{H^0} = \|u\|_{L^2} + \|u\|_{C_0} \quad \text{and} \quad \|u\|_H = \|u\|_{L^2_\mu} + \|u\|_{C_{0,\mu}},$$

respectively. Domains of linear operators L acting in H^0 or in H are, respectively,

$$D^0 = W^{2,2}(\Omega) \cap D^q_0 \quad \text{and} \quad D = W^{2,2}_\mu(\Omega) \cap D^q_{0,\mu},$$

where

$$D^q_0 = \{u,\ u \in C_0(\bar{\Omega}),\ Lu \in C_0(\bar{\Omega}),\ u \in W^{2,q}_{loc}(\Omega),\ q > m\}$$

and

$$D^q_{0,\mu} = \{u,\ u \in C_{0,\mu}(\bar{\Omega}),\ Lu \in C_{0,\mu}(\bar{\Omega}),\ u \in W^{2,q}_{loc}(\Omega),\ q > m\}.$$

The weight function $\mu(x_1)$ is a sufficiently smooth function equal to 1 for $x_1 \le 0$ and $\exp(\nu x_1)$ for $x_1 \ge 1$. Here $\nu > 0$.

We specify assumptions on $F(u,x')$. We suppose that it is a bounded continuous vector-valued function defined for $u \in R^n$, $x' \in \Omega'$. Its derivative $F'_u(u,x')$ with respect to u is a uniformly continuous matrix-valued function satisfying the

Lipschitz condition with respect to u uniformly in x' in any bounded domain in R^n.

Consider the operator linearized about the stationary solution:

$$Lv = a(x')\Delta v + c\frac{\partial v}{\partial x_1} + \sum_{j=1}^{m} b_j(x')\frac{\partial v}{\partial x_j} + F'(w(x), x')v,$$
$$v(x) = 0, \quad x \in \partial\Omega$$

acting in H with the domain D. We note that the operator T of multiplication by μ is a bounded operator from H to H^0 with a bounded inverse. Hence the operator $\tilde{L} = TLT^{-1}$ acts in H^0.

We consider the following eigenvalue problems:

$$a(x')\Delta' u + \sum_{i=2}^{n} b_i(x')\frac{\partial u}{\partial x_i} + (-\xi^2 a(x') + i\xi(c + b_1(x')) + c_-(x'))u = \lambda u, \tag{3.6}$$

$$u(x') = 0, \quad x' \in \partial\Omega', \tag{3.7}$$

and

$$\begin{aligned} a(x')\Delta' u + \sum_{i=2}^{n} b_i(x')\frac{\partial u}{\partial x_i} + [-\xi^2 a(x') + i\xi(c + b_1(x') - 2\nu a(x')) \\ + \nu^2 a(x') - \nu(c + b_1(x')) + c_+(x')]u = \lambda u, \end{aligned} \tag{3.8}$$

$$u(x') = 0, \quad x' \in \partial\Omega', \tag{3.9}$$

where

$$c_\pm(x') = F'(w_\pm(x'), x').$$

We denote them P^- and P^+, respectively.

If the operator $\tilde{L}$ is considered as acting in L^2 with domain $W^{2,2}$ or in C^δ with domain $C^{2+\delta}$ with corresponding boundary conditions, then its essential spectrum is given by all eigenvalues of these problems for all real ξ (see [15], [16], [19]).

We note that the spectrum of the problem P^- is connected with the spectrum of the operator L^- given by (1.7). Since we assume that all eigenvalues of L^- are in the left half-plane, then the spectrum of the problem P^- is also in the left half-plane if $b_1(x') \equiv b_0 I$, where b_0 is a constant and I is the identity matrix. If the matrix b_1 is not scalar, it is probably possible that a part of the spectrum of the problem P^- is in the right half-plane. So we assume that the whole spectrum of this problem is in the half-plane $\operatorname{Re}\lambda < 0$ for all real ξ.

We recall that the operator L^+ has eigenvalues in the right half-plane. In the weighted space, the essential spectrum corresponding to the problem P^+ is moved to the left (cf. [12]). We assume that a positive number ν can be chosen such that all eigenvalues of the problem P^+ are in the left half-plane for all real ξ.

LEMMA 3.1: *Consider the operators $Au = \tilde{L}u$ with the domain $D(A) = \{u : u \in W^{2,2}(\Omega),\ u = 0,\ x \in \partial\Omega\}$ and $\hat{A}u = \tilde{L}u$ with the domain $D(\hat{A}) = \{u : u \in C_\sigma^{2+\delta}(\bar{\Omega}),\ u = 0,\ x \in \partial\Omega\}$, where $C_\sigma^{2+\delta}(\bar{\Omega})$ is a weighted Hölder space with weight $\sigma(x_1) = 1 + x_1^2$. Suppose that both operators A and $\hat{A}$ are Fredholm. If*

$$Au = f, \quad u \in D(A), \quad f \in C_\sigma^\delta(\bar{\Omega}),$$

then $u \in D(\hat{A})$ and $\hat{A}u = f$.

Proof: Let $u_k,\ k = 1, \ldots, p$ be a complete system of solutions of the equation $Au = 0$ and $v_k \in L^2,\ k = 1, \ldots, p$ be a complete system of functions orthogonal to the image of A. Let g_k and $h_k,\ k = 1, \ldots, p$ be smooth finitary functions biorthogonal to u_k and v_k, respectively.

Consider the finite-dimensional operator

$$Ky = (g_1, y)h_1 + \cdots + (g_p, y)h_p,$$

where $(\ ,\ )$ denotes the inner product in L^2. We set $B = A + K$. Then the equation $Bu = 0$ has only zero solution.

Consider the equation $By = f + Ky$. By the Fredholm property it has a solution $y = \tilde{y} \in D(\hat{A})$. On the other hand, $y = u$ is also its solution. Then by uniqueness $u = \tilde{y}$. The lemma is proved. ∎

LEMMA 3.2: *Let G be the maximal region of the complex plane which contains positive real half-axis and does not contain eigenvalues of the problems P^+ and P^- for any real ξ.*

If $\lambda \in G$ is a regular point of the operator $\tilde{L}$ as acting in $L^2(\Omega)$, then it is also a regular point of this operator as acting in H^0.

Proof: Let λ be a regular point of the operator $\tilde{L}$ as acting in $L^2(\Omega)$. Consider the equation

$$\tilde{L}u - \lambda u = f. \tag{3.10}$$

If $f \in H^0$ then $u \in W^{2,2}(\Omega)$. We should show that $u \in C_0(\bar{\Omega})$.

Let $f_n \in L^2(\Omega) \cap C^\delta_\sigma(\bar\Omega)$, where $C^\delta_\sigma(\bar\Omega)$ is a weighted Hölder space; the weight function σ can be taken, for example, as $\sigma(x_1) = x_1^2 + 1$. Since $\lambda \in G$, then equation (3.10) is uniquely solvable in $W^{2,2}(\Omega)$. The operator $\tilde L - \lambda$ is Fredholm as acting in $C^\delta_\sigma(\bar\Omega)$ with the domain $C^{2+\delta}_\sigma(\bar\Omega)$ and its index is zero [19]. The dimension of its kernel is zero because $C^{2+\delta}_\sigma(\bar\Omega) \subset W^{2,2}(\Omega)$. So (3.10) is uniquely solvable in $C^{2+\delta}_\sigma(\bar\Omega)$. Hence its solutions $u_n \in W^{2,2}(\Omega) \cap C^{2+\delta}_\sigma(\bar\Omega)$. Then $u_n \in W^{2,p}(\Omega)$, $p \ge 2$.

The sequence $\{f_n\}$ can be chosen in such a way that it converges to f in $L^2(\Omega)$ and in $C(\bar\Omega)$. Then this convergence occurs also in $L^p(\Omega)$, $p \ge 2$ and this sequence is uniformly bounded in $L^p(\Omega)$. We show that the sequence of solutions $\{u_n\}$ is uniformly bounded in $W^{2,p}(\Omega)$. For this we use the Agmon–Douglis–Nirenberg estimates [1]:

$$\|u_n\|_{W^{2,p}} \le c(\|u_n\|_{L^p} + \|f_n\|_{L^p}).$$

Since the functions f_n are uniformly bounded in L^2 and λ is a regular point of the operator $\tilde L$ as acting in L^2, then the functions u_n are uniformly bounded in $W^{2,2}$. Starting with $p = 2$ and using embedding theorems, by the standard arguments, after a finite number of steps, we obtain that u_n is uniformly bounded in $W^{2,p}(\Omega)$ for any given p. Hence $u_n \in C^{1+\delta}(\bar\Omega)$. From this and convergence $u_n \to u$ in $W^{2,2}$ it follows that $u \in C^1(\bar\Omega)$.

It remains to note that since $u \in W^{2,2}(\Omega) \cap C^1(\bar\Omega)$, then $u(x) \to 0$ as $|x| \to \infty$ and $u \in C_0(\bar\Omega)$. The lemma is proved. ∎

THEOREM 3.3: *Suppose that the problem P^- has all eigenvalues in the left half-plane for all real ξ and there exists ν such that the problem P^+ has all eigenvalues in the left half-plane also for all real ξ.*

If the derivative of the wave $\partial w(x)/\partial x_1$ belongs to the space H, then it is asymptotically stable with shift in this space.

Proof: We should verify that Assumptions 1–3 of the previous subsection are satisfied. The derivative $v = \partial w/\partial x_1$ of the travelling wave $w(x)$ satisfies the problem

$$Lv = 0, \quad x \in \Omega, \quad v = 0, \quad x \in \partial\Omega.$$

Denote $u = Tv$. Then $u \in H^0$ and $\tilde L u = 0$. By estimates of solutions of elliptic problems we obtain $u \in C^1(\bar\Omega)$. From this and the fact that $u \in L^2(\Omega)$, multiplying the equality $\tilde L u = 0$ by u and integrating by parts, we obtain that $u \in W^{1,2}(\Omega)$. By usual arguments we get $u \in W^{2,2}(\Omega)$. From Lemma 3.1, $u \in C^{2+\delta}_\sigma(\bar\Omega)$. Assumptions 1 and 2 follow from this.

Consider Assumption 3. It is known that the operator $\tilde{L}$ with the Dirichlet boundary conditions is a generator of the analytic semigroup in L^2 and C (see [13]). So it is a generator in the space H^0. It follows that the operator L with the Dirichlet conditions is a generator in the space H.

Consider now the location of the spectrum. The spectrum of operator $\tilde{L}$ in the space $L^2(\Omega)$ satisfies conditions of Assumption 3(b) (see [18]). From Lemmas 3.1 and 3.2 it follows that it is also true in the space H^0. Therefore it is true for operator L in the space H. The theorem is proved. ∎

We study now the case where the derivative of the wave does not belong to the space H. We begin with some auxiliary results. We consider the operator

$$\tilde{L}u = \tilde{a}(x)\Delta u + \sum_{j=1}^{m} \tilde{b}_j(x)\frac{\partial u}{\partial x_j} + \tilde{c}(x)u$$

in the domain $\Omega_N \subset \Omega \cap \{x_1 \geq N\}$ such that its boundary is sufficiently smooth and coincides with $\partial\Omega$ for $x_1 \geq N+1$. We represent $\partial\Omega_N = \Gamma_1 \cup \Gamma_2$, where Γ_1 is the part of the boundary in $\partial\Omega$ and Γ_2 is the remaining part.

We assume that the coefficients of the operator are Hölder continuous and have limits at infinity and use the superscript $\pm$ to denote them. The convergence to the limiting coefficients is in the Hölder norm with respect to the variables x'.

As above, the matrices $\tilde{a}(x)$ and $\tilde{b}_j(x)$ are diagonal; $\tilde{c}(x)$ has nonnegative off-diagonal elements. We denote $C_0^{2+\delta}(\bar{\Omega}_N)$ the space of functions from $C^{2+\delta}(\bar{\Omega}_N)$ equal to 0 on $\partial\Omega_N$. We consider the operator $\tilde{L}$ not only on functions from this space but also on functions which do not belong to it. In this case we should understand $\tilde{L}u = 0$ as a local equality in a neighbourhood of each point of Ω.

Lemma 3.3: *Let the operator $\tilde{L}$ as acting from $C_0^{2+\delta}(\bar{\Omega}_N)$ to $C^{\delta}(\bar{\Omega}_N)$ have the essential spectrum in the left half-plane and be invertible.*

Let further $\tilde{L}u = 0$, $\tilde{L}v = 0$ in Ω_N, $u = v = 0$ on Γ_1, $v \geq u$ on Γ_2, $v > 0$ in Ω_N and $u \to 0$ as $x_1 \to +\infty$. Then $v \geq u$ in $\bar{\Omega}_N$.

Proof: The essential spectrum of the operator $\tilde{L}$ is determined by the eigenvalues of the problem

$$\tilde{a}^+(x')\Delta' u + \sum_{j=2}^{m} \tilde{b}_j^+(x')\frac{\partial u}{\partial x_j} + (-\tilde{a}^+(x')\xi^2 + \tilde{b}_1^+(x')i\xi + \tilde{c}^+(x'))u = \lambda u,$$

$$u = 0, \quad x' \in \partial\Omega'$$

in the section of the cylinder for all real ξ. In particular, for $\xi = 0$ all eigenvalues of this problem lie in the left half-plane. Then the problem

$$\tilde{a}^+(x')\Delta' w + \sum_{j=2}^{m} \tilde{b}_j^+(x')\frac{\partial w}{\partial x_j} + \tilde{c}^+(x')w = 0,$$
$$w = -\epsilon, \quad x' \in \partial\Omega'$$

is uniquely solvable, and by virtue of the positiveness theorem its solution is negative in $\bar{\Omega}'$. We denote it $w_\epsilon(x')$.

Consider further the problem

$$\tilde{L}w = 0, \quad w = -\epsilon \quad \text{on } \partial\Omega_N.$$

Existence of its solution follows from the invertibility of the operator $\tilde{L}$. We denote it $\tilde{w}_\epsilon(x)$. Then

$$\tilde{w}_\epsilon(x) \to w_\epsilon(x'), \quad x_1 \to +\infty$$

and this convergence is uniform in x'. We put $u_\epsilon = u + \tilde{w}_\epsilon$. We have

$$\tilde{L}u_\epsilon = 0 \text{ in } \Omega_N,\ u_\epsilon = -\epsilon \ \text{ on } \Gamma_1,\ u_\epsilon \to w_\epsilon(x') < 0,\ x_1 \to +\infty,\ u_\epsilon < v \ \text{ on } \Gamma_2.$$

We show that

$$u_\epsilon < v, \quad x \in \bar{\Omega}_N. \tag{3.11}$$

Indeed, this inequality is satisfied for $x_1 \geq x_1^*$ where x_1^* is sufficiently large, and in a neighbourhood of $\partial\Omega_N$ for $x_1 \leq x_1^*$. Since v is positive in Ω_N, then for some $\tau_0 \geq 1$, $u_\epsilon < \tau_0 v$ in $\bar{\Omega}_N$. We now decrease τ from τ_0 to 1. If for some τ and $x_0 \in \bar{\Omega}_N$, $u_\epsilon(x_0) = \tau v(x_0)$, then $x_0 \in \Omega_N$ and we obtain a contradiction with the positiveness theorem. Thus (3.11) is proved.

In the limit as $\epsilon \to 0$ we obtain $u \leq v$ in $\bar{\Omega}_N$. The lemma is proved. ∎

Remarks: (1) If the matrix $\tilde{c}(x)$ is functionally irreducible and $v(x) \not\equiv u(x)$, then $v(x) > u(x)$ in Ω_N.

(2) We do not require that $v(x) \in C_0^{2+\delta}(\bar{\Omega}_N)$. In particular, it can be unbounded. In this case the operator $\tilde{L}$ is considered not necessarily on functions from $C_0^{2+\delta}(\bar{\Omega}_N)$.

(3) If the coefficients of the operator $\tilde{L}$ do not depend on x_1, then the invertibility condition can be omitted. Indeed, in this case we can put $\tilde{w}_\epsilon(x) \equiv w_\epsilon(x')$.

This function satisfies equation $\tilde{L}w = 0$ in Ω_N, equals $-\epsilon$ on Γ_1, and is negative on Γ_2.

We introduce the following limiting operators:

$$\tilde{L}^{\pm}u = \tilde{a}^{\pm}(x)\Delta u + \sum_{j=1}^{m} \tilde{b}_j^{\pm}(x)\frac{\partial u}{\partial x_j} + \tilde{c}^{\pm}(x)u.$$

LEMMA 3.4: *Let the operator $\tilde{L}$ as acting from $C_0^{2+\delta}(\bar{\Omega})$ to $C^{\delta}(\bar{\Omega})$ have the essential spectrum in the left half-plane, the operator $\tilde{L}^+$ be invertible from $C_0^{2+\delta}(\bar{\Omega}_0^+)$ to $C^{\delta}(\bar{\Omega}_0^+)$, and the operator $\tilde{L}^-$ be invertible from $C_0^{2+\delta}(\bar{\Omega}_0^-)$ to $C^{\delta}(\bar{\Omega}_0^-)$. Here, $\Omega_0^+ = \Omega_N$ for $N = 0$; Ω_0^- is symmetric to Ω_0^+ with respect to $x_1 = 0$.*

Suppose that $\tilde{L}v = 0$ in Ω, $v > 0$ in Ω, $v = 0$ on $\partial\Omega$. Then there is no solution of the equation $\tilde{L}u = 0$ in $C_0^{2+\delta}(\bar{\Omega})$ such that $u \not\equiv 0$, $u \not\equiv v$ and $u \to 0$ as $x_1 \to \pm\infty$.

Proof: For N sufficiently large the operator $\tilde{L}$ is invertible from $C_0^{2+\delta}(\bar{\Omega}_N^+)$ to $C^{\delta}(\bar{\Omega}_N^+)$ and from $C_0^{2+\delta}(\bar{\Omega}_N^-)$ to $C^{\delta}(\bar{\Omega}_N^-)$. So we can apply the previous lemma for it and for these domains.

Suppose that there exists a solution $u(x) \in C_0^{2+\delta}(\bar{\Omega})$ of the equation $\tilde{L}u = 0$, $u \not\equiv 0$. Without loss of generality we can assume that at least one of its components has positive values in the domain $\Omega^0 = \Omega/(\Omega_N^+ \cup \Omega_N^-)$. Otherwise we can change u by $-u$.

Denote $z = \tau v - u$. Then for τ sufficiently large, $z > 0$ in Ω^0 and $\partial z/\partial\nu < 0$ on $\partial\Omega^0 \cap \partial\Omega$. Denote τ_0 the minimal value of τ for which z still satisfies these properties. Clearly $\tau_0 > 0$.

Thus for $\tau = \tau_0$, $z(x) \geq 0$ in $\bar{\Omega}^0$ and (a) $z(x_0) = 0$ for some $x_0 \in \bar{\Omega}^0 \cap \Omega$ or (b) $\partial z/\partial\nu = 0$ at some point on $\partial\Omega^0 \cap \partial\Omega$. Then from the previous lemma $z(x) \geq 0$ in $\bar{\Omega}$. In case (a) we obtain a contradiction with the positiveness theorem, and in case (b) with the negativeness of the normal derivative on the boundary. The theorem is proved. ∎

Remark: For the conditions of the lemma it can be shown similarly that the operator $\tilde{L}$ cannot have positive eigenvalues in $C_0^{2+\delta}(\bar{\Omega})$.

LEMMA 3.5: *Let the operator $\tilde{L}^+$ as acting from $C_0^{2+\delta}(\bar{\Omega}_0^+)$ to $C^{\delta}(\bar{\Omega}_0^+)$ have the essential spectrum in the left half-plane. Then it does not have zero eigenvalue.*

Proof: We apply Remark 3 to Lemma 3.3. For this we construct a positive solution v_+ of the inequality $\tilde{L}^+v \leq 0$ which is an upper function and provides existence of a positive solution $v(x)$ of the equation $L^+v = 0$. We put

$$v_+(x) = e^{-\kappa x_1}\phi_+(x')$$

for some small positive κ. The function ϕ_+ should be positive in Ω' and satisfy the inequality

$$\tilde{a}^+(x')\Delta'\phi_+ + \sum_{j=2}^{m}\tilde{b}_j^+(x')\frac{\partial\phi_+}{\partial x_j} + (\tilde{a}^+(x')\kappa^2 - \tilde{b}_1^+(x')\kappa + \tilde{c}^+(x'))\phi_+ \le 0.$$

If κ is sufficiently small, we can take as ϕ_+ the principal eigenfunctions of the operator corresponding to the left hand side of this inequality.

Suppose that there is a solution $u(x) \in C_0^{2+\delta}(\bar{\Omega}_0^+)$ of the equation $\tilde{L}u = 0$. Then $u(x) \to 0$ as $x_1 \to +\infty$. Otherwise, $\lambda = 0$ would be a point of the essential spectrum of this operator [19]. Since $v(x) > 0$ in Ω_0^+ and on Γ_2, we can apply Remark 3 to the functions τv and u for any positive τ. If $\tau \to 0$, we obtain in the limit $u \le 0$ in $\bar{\Omega}_0^+$. Changing u to $-u$ we show that $u \ge 0$. Thus $u \equiv 0$. The lemma is proved. ∎

Similarly, we show the invertibility of the operator $\tilde{L}^-$ in $C_0^{2+\delta}(\bar{\Omega}_0^-)$.

The last theorem states the asymptotic stability without shift when the derivative of the wave does not belong to the weighted space.

THEOREM 3.4: *Suppose that the problem P^- has all eigenvalues in the left half-plane for all real ξ and there exists ν such that the problem P^+ has all eigenvalues in the left half-plane also for all real ξ.*

If the derivative of the wave $\partial w(x)/\partial x_1$ does not belong to the space H, then the wave is asymptotically stable in this space.

Proof: We should verify that all eigenvalues of the operator L in the space H lie in the left half-plane. Consider the operator $\hat{L} = T^{-1}LT$ in the space H^0, where T is the operator of multiplication by μ. It is sufficient to verify that all its eigenvalues lie in the left half-plane.

The essential spectrum of this operator is determined by the problems (3.6), (3.7) and (3.8), (3.9) and lies in the left half-plane. Suppose that it has an eigenvalue with nonnegative real part. Then its principal eigenvalue, being real, is nonnegative and tends uniformly to zero as $x_1 \to \pm\infty$. We can apply Lemma 3.4 since the operators

$$\hat{L}^+u = a^+(x')\Delta u + \sum_{j=2}^{m} b_j^+(x')\frac{\partial u}{\partial x_j} + (b_1^+(x') + c - 2a^+(x')\nu)\frac{\partial u}{\partial x_1}$$
$$+(a^+(x')\nu^2 - (b_1^+(x') + c)\nu + c^+(x'))u$$

and

$$\hat{L}^{-}u = a^{-}(x^{'})\Delta u + \sum_{j=2}^{m} b_j^{-}(x^{'})\frac{\partial u}{\partial x_j} + (b_1^{+}(x^{'}) + c)\frac{\partial u}{\partial x_1} + c^{-}(x^{'})u$$

are invertible (i.e. they do not have zero eigenvalue) as acting from $C_0^{2+\delta}(\bar{\Omega}_N^{+})$ to $C^{\delta}(\bar{\Omega}_N^{+})$ and from $C_0^{2+\delta}(\bar{\Omega}_N^{-})$ to $C^{\delta}(\bar{\Omega}_N^{-})$, respectively (Lemma 3.5).

Thus we have shown that the equation $\hat{L}u = \lambda u$ with nonnegative λ cannot have nonzero solutions converging to zero at infinity and different from the function $v = -\partial w/\partial x_1\ \mu$. Hence the corresponding eigenfunctions cannot belong to the space H^0. Otherwise it would converge to zero at infinity and, consequently, coincide with v. This contradicts the assumption that the derivative of the wave does not belong to the weighted space. The theorem is proved. ∎

References

[1] S. Agmon, A. Douglis and L. Nirenberg, *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions*, Communications on Pure and Applied Mathematics **12** (1959), 623–727.

[2] H. Berestycki, B. Larrouturou and J. M. Roquejoffre, *Stability of travelling fronts in a model for flame propagation. Part I: Linear analysis*, Archive for Rational Mechanics and Analysis **117** (1992), 97–117.

[3] H. Berestycki and L. Nirenberg, *Travelling fronts in cylinders*, Annales de l'Institut Henri Poincaré. Analyse Non Linéaire **9** (1992), 497–572.

[4] R. Gardner, *Existence of multidimensional travelling wave solutions of an initial-boundary value problem*, Journal of Differential Equations **61** (1986), 335–379.

[5] D. Gilbarg and N. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag, New York, 1983.

[6] S. Heinze, *Traveling waves for semilinear parabolic partial differential equations in cylindrical domains*, Preprint No. 506, Heidelberg, 1989, 46 pp.

[7] A. N. Kolmogorov, I. G. Petrovsky and N. S. Piskunov, *A study of the diffusion equation with increase in the amount of substance, and its application to a biological problem*, Bulletin of Moscow University, Mathematics Mechanics **1:6** (1937), 1–26; in *Selected Works of A.N. Kolmogorov*, Vol. 1 (V. M. Tikhomirov, ed.), Kluwer, London, 1991.

[8] O. A. Ladyzhenskaya, V. A. Solonnikov and N. N. Uraltseva, *Linear and Quasilinear Equations of Parabolic Type*, Nauka, Moscow, 1967.

[9] O. A. Ladyzhenskaya and N. N. Uraltseva, *Linear and Quasilinear Equations of Elliptic Type,* Nauka, Moscow, 1973.

[10] A. Lunardi, *Analytic Semigroups and Optimal Regularity in Parabolic Problems,* Birkhäuser, Boston, 1995.

[11] J. F. Mallordy and J. M. Roquejoffre, *A parabolic equation of KPP type in higher dimensions,* SIAM Journal of Mathematical Analysis **26** (1995), 1–20.

[12] D. H. Sattinger, *On the stability of waves of nonlinear parabolic systems,* Advances in Mathematics **22** (1976), 312–355.

[13] H. B. Stewart, *Generation of analytic semigroups by strongly elliptic operators,* Transactions of the American Mathematical Society **199** (1974), 141–162.

[14] J. M. Vega, *Traveling wavefronts of reaction-diffusion equations in cylindrical domains,* Communications in Partial Differential Equations **18** (1993), 505–531.

[15] A. I. Volpert, Vit. A. Volpert and Vℓ. A. Volpert, *Traveling Wave Solutions of Parabolic Systems,* Translation of Mathematical Monographs, Vol. 140, American Mathematical Society, Providence, 1994.

[16] V. A. Volpert, *On the spectrum of an ellitpic operator in an unbounded cylindrical domain,* Doklady Adakemii Nauk Ukrainskoi SSR, Seriya A (1981), No. 12, pp. 9–12.

[17] V. A. Volpert and A. I. Volpert, *Travelling wave solutions of monotone parabolic systems,* Preprint, Institute of Chemical Physics, Chernogolovka, 1990 (Russian); Preprint No. 146, Université Lyon 1, Villeurbanne, 1993.

[18] V. A. Volpert and A. I. Volpert, *Spectrum of elliptic operators and stability of travelling waves,* Preprint No. 237, Université Lyon 1, 1996.

[19] V. Volpert, A. Volpert and J. F. Collet, *Topological degree for elliptic operators in unbounded domains,* Advances in Differential Equations, to appear.